Lagrangian capacity of symplectic manifolds

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These are notes from my talk in the symplectic geometry seminar in the working group of Klaus Mohnke, Chris Wendl, and Thomas Walpuski in Berlin.

The Lagrangian capacity is a symplectic capacity defined by Cieliebak and Mohnke. In this talk, via a neck-stretching argument of Cieliebak-Mohnke [2], we compare the Lagrangian and McDuff-Seigel capacities of Liouville domains. As a consequence, we show that the Lagrangian capacity of a 4-dimensional convex toric domain is equal to its diagonal. This positively settles a conjecture of Cieliebak and Mohnke for the Lagrangian capacity of the 4-dimensional ellipsoids.

Our main references are Cieliebak-Mohnke [2], Pereira [5], Rizell [3].

1 Recall from the previous talk

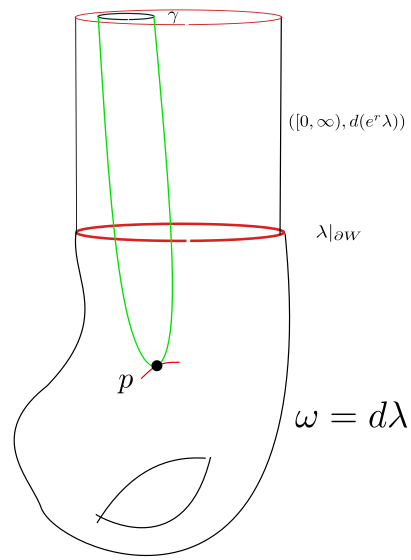
Theorem 1.1. *(Cieliebak-Mohnke [2], 2014) There are exactly* $(n-1)!$ *holomorphic sphere in the homology class of line* $[\mathbb{CP}^1]$ *in* $\mathbb{CP}^n$ *passing through a generic point* $p \in \mathbb{CP}^n$ *and having a tangency order* $n-1$ *to generic local divisor containing* p*. In terms of the notations from my previous talk, this means*

$$N_{\mathbb{CP}^n,[\mathbb{CP}^1]} \ll \mathcal{T}^{n-1}p \gg= (n-1)!$$

Definition 1.2. (McDuff-Siegel Capacities [4], 2022) Let (W, λ) be a non-degenerated Liouville domain. Let D_p be a smooth local symplectic divisor passing through $p \in \operatorname{Int} X$. Define $\mathcal{J}(\widehat{W}, D_p)$ to be the space of all admissible almost complex structures on the symplectic completion $\widehat{W}$ that are integrable near p and D_p is holomorphic. For $k \in \mathbb{N}$, define

$$\mathrm{MS}^1_k(W) := \sup_{J \in \mathcal{J}(\widehat{W}, D_p)} \inf_u \int_{D^2} u^* d\lambda \in [0, \infty]$$

where the infimum is taken over asymptotic J-holomorphic disks in $\widehat{W}$ with $\ll \mathcal{T}^{k-1}p \gg$.



2 Toric domains

Definition 2.1. (Convex toric domains) Let $\Omega \subset \mathbb{R}^n_+$ denotes a convex subset containing 0. Consider $\mu : \mathbb{C}^n \to \mathbb{R}^n_+$ given by

$$\mu(z_1, z_2, \ldots, z_n) := \pi(|z_1|^2, \ldots, |z_n|^2).$$

A convex toric domain in $\mathbb{C}^n$ is a subset $X_\Omega \subset \mathbb{C}^n$ of the form

$$X_\Omega = \mu^{-1}(\Omega).$$

The diagonal of a convex toric domain X_Ω is defined by

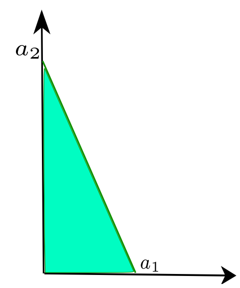
$$\text{diagonal}(X_\Omega) := \sup\{a > 0 : (a, a, \ldots, a) \in \Omega\}.$$

Remark 2.2. Let $\delta = \text{diagonal}(X_\Omega)$, the Lagrangian torus $\mathbb{S}^1(\sqrt{\frac{\delta}{\pi}}) \times \cdots \times \mathbb{S}^1(\sqrt{\frac{\delta}{\pi}})$ stands on the boundary of (X_Ω, ω_0).

Example 2.3. (Example of a convex toric domain) Let $0 < a_1 \leq a_2 < \infty$,

$$E(a_1, a_2) := \{(z_1, z_2) \in \mathbb{C}^2 : \sum_{i=1}^{2} \frac{\pi |z_i|^2}{a_i} \leq 1\}$$

$$E(a_1, a_2) := \mu^{-1}(\Omega)$$



$$\text{diagonal}(E(a_1, a_2)) = \frac{1}{\sum_{i=1}^{2} \frac{1}{a_i}}.$$

In general,

$$\text{diagonal}(E(a_1, \ldots, a_n)) := \frac{1}{\sum_{i=1}^{n} \frac{1}{a_i}}.$$

Definition 2.4. (Symplectic engery of Langrangain tori) Let (M, ω) be a symplectic manifold and $L \subset (M, \omega)$ be an embedded Lagrangian torus. The symplectic energy of L is defined as

$$\mathrm{A}_{\min}(L) := \inf\{\int_{D^2} u^*\omega > 0 : u : (D^2, \partial D^2) \to (M, L)\}.$$

Example 2.5. For the Clifford torus $L_{Clif} := \mathbb{S}^1(r) \times \cdot \times \mathbb{S}^1(r) \subset (\mathbb{C}^n, \omega_0)$ we have

$$\mathrm{A}_{\min}(L_{Clif}) = \pi r^2$$

Example 2.6. For the product torus $L_{std} := \mathbb{S}^1(1) \times \mathbb{S}^1(2) \subset (\mathbb{C}^2, \omega_0)$ we have

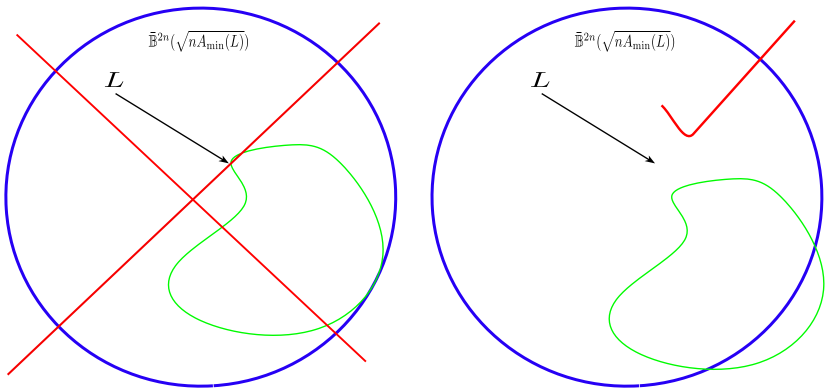
$$\mathrm{A}_{\min}(L_{std}) = \pi.$$

Example 2.7. Let $\delta = \text{diagonal}(X_\Omega)$, $L_{std} = \mathbb{S}^1(\sqrt{\frac{\delta}{\pi}}) \times \cdots \times \mathbb{S}^1(\sqrt{\frac{\delta}{\pi}}) = \mu^{-1}(\delta, \delta) \subset (X_\Omega, \omega_0)$, so

$$\text{diagonal}(X_\Omega) = A_{min}(L_{std}).$$

The energy of a Lagrangian sub-manifold remembers how the Lagrangian sits in the ambient symplectic manifold. This was conjectured by Cieliebak and Mohnke as follows:

Conjecture 2.8. *(Cieliebak-Mohnke [2], Conjecture 1.9) If a Lagrangian torus in* $(\mathbb{C}^n, \omega_0)$ *intersects the interior of the ball* $\bar{\mathbb{B}}^{2n}(\sqrt{n\,\mathrm{A}_{\min}(L)/\pi})$, *then it must intersect its exterior as well.*



Conjecture 2.9. *(Weak version of this conjecture) A Hamiltonian flow on* $(\mathbb{C}^n, \omega_0)$ *can not squeeze a Lagrangian torus* L *in* $\mathbb{C}^n$ *into the open ball of radius* $\sqrt{nA_{\min}(L)/\pi}$.

Theorem 2.10. *(Georgios Rizell [3]) The following is true:*

- *Every Lagrangian torus in* $(\bar{\mathbb{B}}^4(1), \omega_0)$ *with energy* $\pi/2$ *lies totally on the boundary* $\mathbb{S}^3$.
- *Every Lagrangian torus in* $(\bar{\mathbb{B}}^4(1), \omega_0)$ *with energy* $\pi/2$ *is Hamiltonian isotopic to the Clifford torus* $\mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^1(\frac{1}{\sqrt{2}})$ *on* $\mathbb{S}^3$.

Open Problem 2.11. The above conjecture is open for $n > 2$.

Definition 2.12. (Lagrangian Capacity, Cieliebak-Mohnke [2], 2014) For a symplectic manifold (M, ω), the number $c_L(M, \omega) \in [0, \infty]$ defined by

$$c_L(M, \omega) := \sup\{\mathrm{A}_{\min}(L) : L \subset (M, \omega) \text{ is embedded Lagrangian torus }\}$$

is a symplectic capacity known as Lagrangian capacity of (M, ω).

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Theorem 2.13. *(Cieliebak-Mohnke [2], 2014)*

$$c_L(\mathbb{CP}^n, \omega_{FS}) = \frac{\pi}{n+1}.$$

$$c_L(\mathbb{B}^{2n}(1), \omega_0) = \frac{\pi}{n}.$$

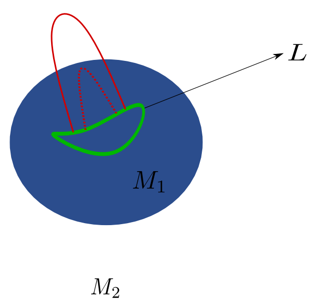
$$c_L(\mathbb{B}^2(1) \times \mathbb{C}^{(n-1)}, \omega_0) = \pi$$

Theorem 2.14. *(Cieliebak-Mohnke [2], 2014) The Lagrangian capacity satisfies*

- $c_L(M, \alpha\omega) = \alpha c_L(M, \omega), \forall \alpha > 0.$
- *If there exists a symplectic embedding* $i : (M_1, \omega_1) \to (M_2, \omega_2)$ *with* $\pi_2(M_2, i(M_1)) = 0$*, then*
$$c_L(M_1, \alpha\omega_1) \le c_L(M_2, \alpha\omega_2).$$
- $0 < \mathrm{c_L}(\mathbb{B}^{2n}(1), \omega_0)$*, and* $0 < c_L(\mathbb{B}^2(1) \times \mathbb{C}^{(n-1)}, \omega_0) < \infty$.

Remark 2.15. In the second bullet point above, the condition $\pi_2(M_2, i(M_1)) = 0$ is very important. For any Lagrangian L in $i(M_1)$, there might be disks in M_2 with boundary on L which has area less than any other disk in M_1 with boundary on L.

$$A^{M_2}_{min}(L) < A^{M_1}_{min}(L)$$



In this case and hence

$$c_L(M_1, \alpha\omega_1) > c_L(M_2, \alpha\omega_2).$$

For example, take $M_1 = \mathbb{B}^{2n}(1)$ and $M_2 = \mathbb{CP}^n$, by the above theorem

$$c_L(\mathbb{CP}^n, \omega_{FS}) < c_L(\mathbb{B}^{2n}(1), \omega_0).$$

Conjecture 2.16. *(Cieliebak-Mohnke [2], 2014)*

$$c_L(E(a_1, a_2, \ldots, a_n), \omega_0) = \frac{1}{\sum_{i=1}^n \frac{1}{a_i}}.$$

Theorem 2.17. *(Pereira [5], 2022) If* (X, λ) *is a Liouville domain, then*

$$c_L(X, d\lambda) \le \inf_k \frac{\mathrm{MS}^1_k(X)}{k},$$

where $\mathrm{MS}^1_k(X)$ *is the kth-McDuff-Siegel capacity of* X*.*

Corollary 2.18. *Let X_Ω be a four dimensional convex toric domain, then*

$$c_L(X_\Omega, \omega_0) = \delta.$$

In particular, $c_L(E(a_1,a_2),\omega_0) = \frac{1}{\sum_{i=1}^2 \frac{1}{a_i}}$.

Proof.

- Let $\delta = \text{diagonal}(X_\Omega)$, $L_{std} = \mathbb{S}^1(\sqrt{\frac{\delta}{\pi}}) \times \cdots \times \mathbb{S}^1(\sqrt{\frac{\delta}{\pi}}) \subset (X_\Omega, \omega_0)$, so

$$\delta = A_{min}(L_{std}) \leq \mathrm{c}_L(X_\Omega, \omega_0).$$

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$$\delta \leq \mathrm{c}_L(X_\Omega,\omega_0) \leq \frac{\mathrm{MS}^1_k(X_\Omega)}{k} \leq \frac{\delta(k+1)}{k} = \delta + \frac{\delta}{k}$$

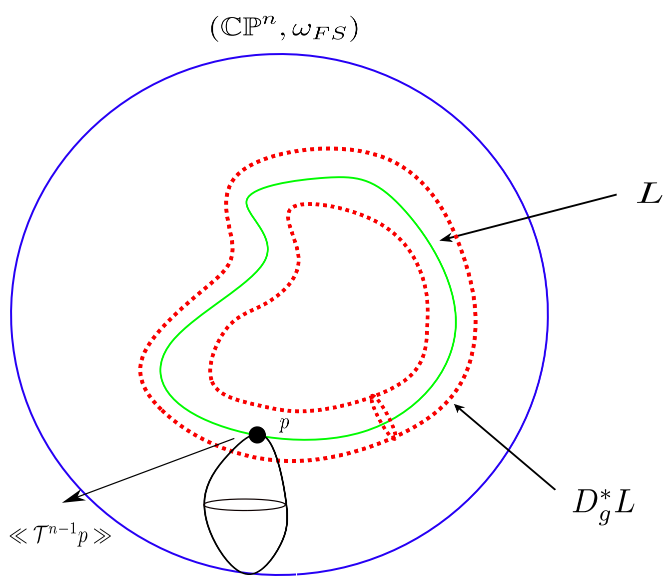
□

Proof. (Proof sketch of $c_L(\mathbb{CP}^n, \omega_{FS}) = \frac{\pi}{n+1}$)
Note that $L_{std} := \mathbb{S}^1(\frac{1}{\sqrt{n+1}}) \times \cdots \times \mathbb{S}^1(\frac{1}{\sqrt{n+1}}) \subset \mathbb{B}^{2n}(1) \subset \mathbb{CP}^n$, so

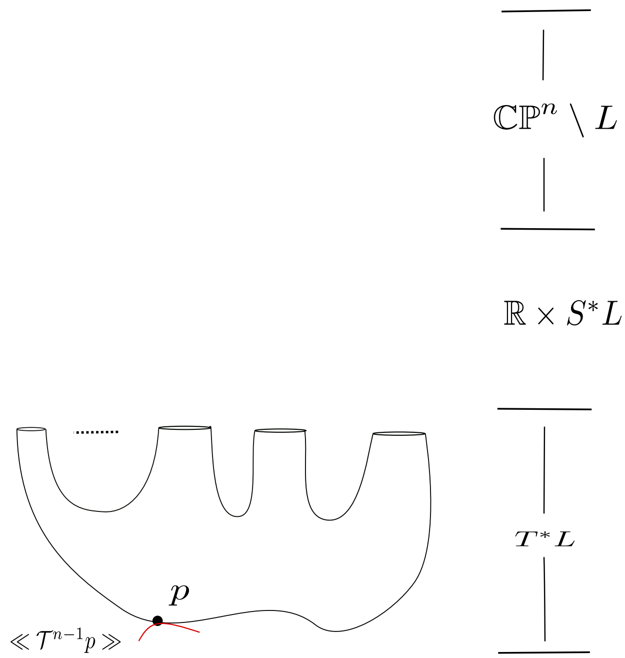
$$\frac{\pi}{n+1} = A_{min}(L_{std}) \leq c_L(\mathbb{CP}^n, \omega_{FS}).$$

To prove: for every Lagrangian torus L in $\mathbb{CP}^n$, there exists a smooth disk : $u : (D^2, \partial D^2) \to (\mathbb{CP}^n, L)$ with

$$0 < \int_{D^2} u^*\omega_{FS} \leq \frac{\pi}{n+1}.$$



There is a holomorphic sphere passing through p with the constraint $\ll \mathcal{T}^{n-1}p \gg$. Stretching the neck of this sphere along $\partial D_g^* L$ leads to a building:



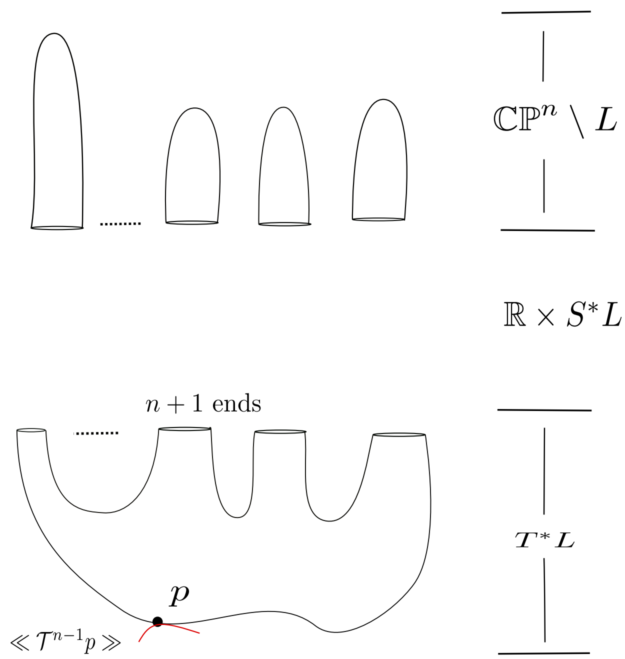
$$\operatorname{ind}(C_{bot}) = (n-3)(2-l) + \sum_{i=1}^{l}(\mathrm{RS}(\gamma_i) + \frac{1}{2}\dim(\gamma_i)) - 2n + 2 - 2(n-1).$$

$$\mathrm{RS}(\gamma_i) = \frac{1}{2}\mathrm{Nullity}(\gamma_i) + \text{Morse-ind}(\gamma_i).$$

$$\mathrm{RS}(\gamma_i) = \frac{1}{2}\dim(\gamma_i) = \frac{1}{2}(n-1).$$

$$\operatorname{ind}(C_{bot}) = 2l - 2n - 2.$$

We must have $l \geq n+1$.



- Denote these disks by $u_1, u_2, \ldots, u_n, u_{n+1}$. We have that

$$\sum_{i=1}^{n+1} \int_{D^2} u_i^* \omega_{FS} \leq \pi.$$

- $$\int_{D^2} u_1^* \omega_{FS} \leq \frac{1}{n+1} \sum_{i=1}^{n+1} \int_{D^2} u_i^* \omega_{FS} \leq \frac{\pi}{n+1}.$$

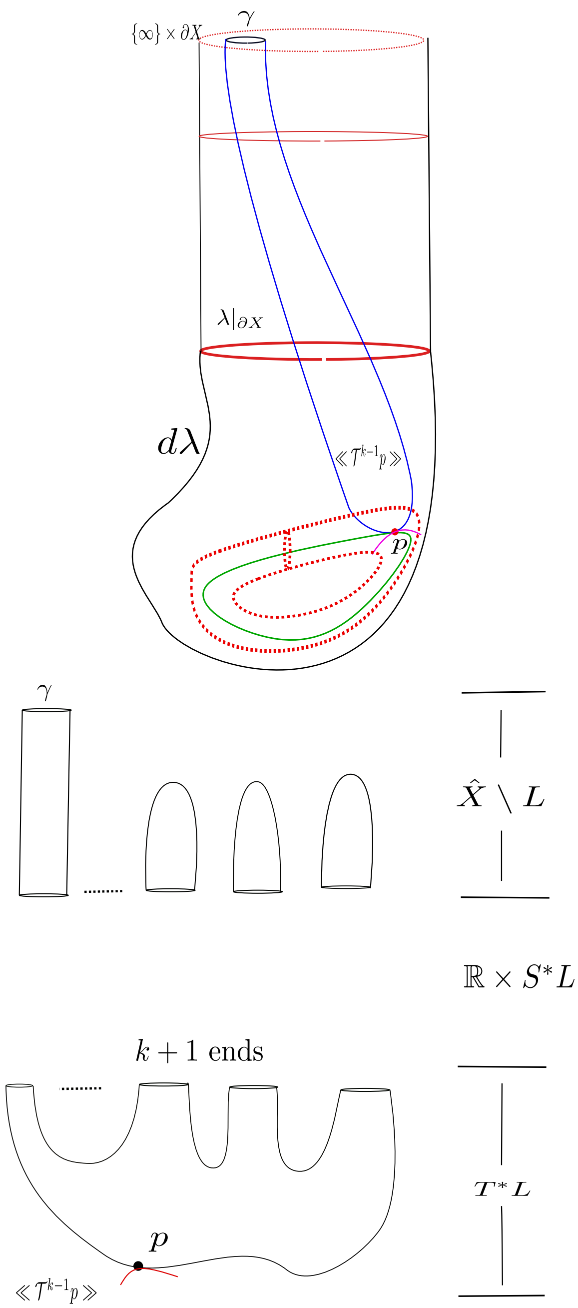
- This proves

$$c_L(\mathbb{CP}^n, \omega_{FS}) = \frac{\pi}{n+1}.$$

$$c_L(\mathbb{B}^{2n}(1), \omega_0) = \frac{\pi}{n}.$$

□

Proof. (Proof sketch of $c_L(X, d\lambda) \leq \inf_k \frac{\mathrm{MS}_k^1(X)}{k}$)



- Denote these disks by $u_1, u_2, \ldots, u_k$. We have that

$$\sum_{i=1}^{k} \int_{D^2} u_i^* \omega_{FS} \leq \mathrm{MS}_k^1(X).$$

- For one of these disks, we have

$$\int_{D^2} u_1^*\omega_{FS} \leq \frac{1}{k}\sum_{i=1}^{k}\int_{D^2} u_i^*\omega_{FS} \leq \frac{\mathrm{MS}_k^1(X)}{k}.$$

- This proves

$$c_L(X, d\lambda) \leq \inf_k \frac{\mathrm{MS}_k^1(X)}{k}.$$

□

References

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